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Chapter

# Existence, Uniqueness and Approximate Solutions of Fuzzy Fractional Differential Equations

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## Abstract

In this paper, the Cauchy problem of fuzzy fractional differential equations

$$T_\gamma u(t) = F(t, u(t)),\ u(t_0) = u_0, \tag{1}$$

with fuzzy conformable fractional derivative ($\gamma$-differentiability, where $\gamma \in (0, 1]$) are introduced. We study the existence and uniqueness of solutions and approximate solutions for the fuzzy-valued mappings of a real variable, we prove some results by applying the embedding theorem, and the properties of the fuzzy solution are investigated and developed. Also, we show the relation between a solution and its approximate solutions to the fuzzy fractional differential equations of order $\gamma$.


**Keywords:** fuzzy conformable fractional derivative, fuzzy fractional differential equations, existence and uniqueness of solution, approximate solutions, Cauchy problem of fuzzy fractional differential equations

## 1. Introduction

In this paper, we will study Fuzzy solutions to

$$T_\gamma u(t) = F(t, u(t)), \quad u(t_0) = u_0,\ \ \gamma \in (0, 1], \tag{2}$$

where subject to initial condition $u_0$ for fuzzy numbers, by the use of the concept of conformable fractional $H$-differentiability, we study the Cauchy problem of fuzzy fractional differential equations for the fuzzy valued mappings of a real variable. Several import-extant results are obtained by applying the embedding theorem in [1] which is a generalization of the classical embedding results [2, 3].

In Section 2 we recall some basic results on fuzzy number. In Section 3 we introduce some basic results on the conformable fractional differentiability [4, 5] and conformable integrability [5, 6] for the fuzzy set-valued mapping in [7]. In Section 4 we show the relation between a solution and its approximate solution to the Cauchy problem of the fuzzy fractional differential equation, and furthermore, and we prove the existence and uniqueness theorem for a solution to the Cauchy problem of the fuzzy fractional differential equation.

## 2. Preliminaries

We now recall some definitions needed in throughout the paper. Let us denote by $\mathrm{R}_{\mathcal{F}}$ the class of fuzzy subsets of the real axis $\{u : \mathrm{R} \to [0, \ 1]\}$ satisfying the following properties:

i. $u$ is normal: there exists $x_0 \in \mathrm{R}$ with $u(x_0) = 1$,

ii. $u$ is convex fuzzy set: for all $x, t \in \mathrm{R}$ and $0 < \lambda \le 1$, it holds that

$$u(\lambda x + (1-\lambda)t) \ge \min\{u(x), u(t)\}, \tag{3}$$

iii. $u$ is upper semicontinuous: for any $x_0 \in \mathrm{R}$, it holds that

$$u(x_0) \ge \lim_{x \to x_0} u(x), \tag{4}$$

iv. $[u]^0 = cl\{x \in \mathrm{R} | u(x) > 0\}$ is compact.

Then $\mathrm{R}_{\mathcal{F}}$ is called the space of fuzzy numbers see [8]. Obviously, $\mathrm{R} \subset \mathrm{R}_{\mathcal{F}}$. If $u$ is a fuzzy set, we define $[u]^\alpha = \{x \in \mathrm{R} | u(x) \ge \alpha\}$ the $\alpha$-level (cut) sets of $u$, with $0 < \alpha \le 1$. Also, if $u \in \mathrm{R}_{\mathcal{F}}$ then $\alpha$-cut of $u$ denoted by $[u]^\alpha = [u_1^\alpha, u_2^\alpha]$.

***Lemma 1*** see [9] *Let* $u, v : \mathrm{R}_{\mathcal{F}} \to [0, 1]$ *be the fuzzy sets. Then* $u = v$ *if and only if* $[u]^\alpha = [v]^\alpha$ *for all* $\alpha \in [0, 1]$.

For $u, v \in \mathrm{R}_{\mathcal{F}}$ and $\lambda \in \mathrm{R}$ the sum $u + v$ and the product $\lambda u$ are defined by

$$[u+v]^\alpha = [u_1^\alpha + v_1^\alpha, u_2^\alpha + v_2^\alpha], \tag{5}$$

$$[\lambda u]^\alpha = \lambda [u]^\alpha = \{ [\lambda u_1^\alpha, \lambda u_2^\alpha], \ \lambda \ge 0; [\lambda u_2^\alpha, \lambda u_1^\alpha], \ \lambda < 0, \tag{6}$$

$\forall \alpha \in [0, 1]$. Additionally if we denote $\hat{0} = \chi_{\{0\}}$, then $\hat{0} \in \mathrm{R}_{\mathcal{F}}$ is a neutral element with respert to $+$.

Let $d : \mathrm{R}_{\mathcal{F}} \times \mathrm{R}_{\mathcal{F}} \to \mathrm{R}_+ \cup \{0\}$ by the following equation:

$$d(u, v) = \sup_{\alpha \in [0,1]} d_H([u]^\alpha, \ [v]^\alpha), \ \textit{forall} \ u, v \in \mathrm{R}_{\mathcal{F}}, \tag{7}$$

where $d_H$ is the Hausdorff metric defined as:

$$d_H([u]^\alpha, \ [v]^\alpha) = \max\{|u_1^\alpha - v_1^\alpha|, \ u_2^\alpha - v_2^\alpha||\} \tag{8}$$

The following properties are well-known see [10]:

$$d(u+w, v+w) = d(u, v) \quad \textit{and} \quad d(u, v) = d(v, u), \quad \forall \ u, v, w \in \mathrm{R}_{\mathcal{F}}, \tag{9}$$

$$d(ku, kv) = |k| d(u, v), \quad \forall k \in \mathrm{R}, \ u, v \in \mathrm{R}_{\mathcal{F}} \tag{10}$$

$$d(u+v, w+e) \le d(u, w) + d(v, e), \quad \forall \ u, v, w, e \in \mathrm{R}_{\mathcal{F}}, \tag{11}$$

and $(\mathrm{R}_{\mathcal{F}}, d)$ is a complete metric space.

***Definition 1*** The mapping $u : [0, a] \to \mathrm{R}_{\mathcal{F}}$ for some interval $[0, a]$ is called a fuzzy process. Therefore, its $\alpha$-level set can be written as follows:

$$[u(t)]^\alpha = [u_1^\alpha(t), u_2^\alpha(t)], \ t \in [0, a], \ \alpha \in [0, 1]. \tag{12}$$

***Theorem 1.1*** [11] Let $u : [0, a] \rightarrow R_{\mathcal{F}}$ *be Seikkala differentiable and denote* $[u(t)]^{\alpha} = [u_1^{\alpha}(t), u_2^{\alpha}(t)]$. *Then, the boundary function* $u_1^{\alpha}(t)$ *and* $u_2^{\alpha}(t)$ *are differentiable and*

$$[u'(t)]^{\alpha} = \left[ \left(u_1^{\alpha}\right)'(t), \quad \left(u_2^{\alpha}\right)'(t) \right], \quad \alpha \in [0, 1]. \tag{13}$$

***Definition 2*** [12] Let $u : [0, a] \rightarrow R_{\mathcal{F}}$. The fuzzy integral, denoted by $\int_b^c u(t)dt, b, c \in [0, a]$, is defined levelwise by the following equation:

$$\left[ \int_b^c u(t)dt \right]^{\alpha} = \left[ \int_b^c u_1^{\alpha}(t)dt, \quad \int_b^c u_2^{\alpha}(t)dt \right], \tag{14}$$

for all $0 \leq \alpha \leq 1$. In [12], if $u : [0, a] \rightarrow R_{\mathcal{F}}$ is continuous, it is fuzzy integrable.

***Theorem 1.2*** [13] If $u \in R_{\mathcal{F}}$, *then the following properties hold*:

i. $$[u]^{\alpha_2} \subset [u]^{\alpha_1}, \textit{ if } \quad 0 \leq \alpha_1 \leq \alpha_2 \leq 1; \tag{15}$$

ii. $\{\alpha_k\} \subset [0, 1]$ *is a nondecreasing sequence which converges to* $\alpha$ *then*

$$[u]^{\alpha} = \bigcap_{k \geq 1} [u]^{\alpha_k}. \tag{16}$$

Conversely if $A_{\alpha} = \{ [u_1^{\alpha}, u_2^{\alpha}]; \ \alpha \in (0, 1] \}$ *is a family of closed real intervals verifying* $(i)$ *and* $(ii)$, *then* $\{A_{\alpha}\}$ *defined a fuzzy number* $u \in R_{\mathcal{F}}$ *such that* $[u]^{\alpha} = A_{\alpha}$.

From [1], we have the following theorems:

***Theorem 1.3*** There exists a real Banach space $X$ *such that* $\mathbb{R}_{\mathcal{F}}$ *can be the embedding as a convex cone* $C$ *with vertex* $0$ *into* $X$. *Furthermore, the following conditions hold*:

i. the embedding $j$ is isometric,

ii. addition in $X$ induces addition in $\mathbb{R}_{\mathcal{F}}$, i.e, for any $u, v \in \mathbb{R}_{\mathcal{F}}$,

iii. multiplication by a nonnegative real number in $X$ induces the corresponding operation in $\mathbb{R}_{\mathcal{F}}$, i.e., for any $u \in \mathbb{R}_{\mathcal{F}}$,

iv. C-C is dense in $X$,

v. $C$ is closed.

## 3. Fuzzy conformable fractional differentiability and integral

***Definition 3*** [4] *Let* $F : (0, a) \rightarrow \mathbb{R}_{\mathcal{F}}$ *be a fuzzy function.* $\gamma^{th}$ *order "fuzzy conformable fractional derivative" of F is defined by*

$$T_{\gamma}(F)(t) = \lim_{\varepsilon \rightarrow 0^+} \frac{F(t + \varepsilon t^{1-\gamma}) \ominus F(t)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0^+} \frac{F(t) \ominus F(t - \varepsilon t^{1-\gamma})}{\varepsilon}. \tag{17}$$

*for all* $t > 0, \gamma \in (0, 1)$. *Let* $F^{(\gamma)}(t)$ *stands for* $T_{\gamma}(F)(t)$. *Hence*

$$F^{(\gamma)}(t) = \lim_{\varepsilon \to 0^+} \frac{F(t + \varepsilon t^{1-\gamma}) \ominus F(t)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{F(t) \ominus F(t - \varepsilon t^{1-\gamma})}{\varepsilon}. \tag{18}$$

*If F is $\gamma$- differentiable in some $(0,\ a)$, and $\lim_{t\to 0^+} F^{(\gamma)}(t)$ exists, then*

$$F^{(\gamma)}(0) = \lim_{t \to 0^+} F^{(\gamma)}(t) \tag{19}$$

*and the limits (in the metric $d$).*

Remark 1 *From the definition, it directly follows that if $F$ is $\gamma$-differentiable then the multivalued mapping $F_\alpha$ is $\gamma$-differentiable for all $\alpha \in [0,1]$ and*

$$T_\gamma F_\alpha = \left[F^{(\gamma)}(t)\right]^\alpha, \tag{20}$$

where $T_\gamma F_\alpha$ is denoted from the conformable fractional derivative of $F_\alpha$ of order $\gamma$. The converse result does not hold, since the existence of Hukuhara difference $[u]^\alpha \ominus [v]^\alpha,\ \ \alpha \in [0,1]$ does not imply the existence of H-difference $u \ominus v$.

Theorem 1.4 [4] Let $\gamma \in (0,1]$.

If $F$ is differentiable and $F$ is $\gamma$-differentiable then

$$T_\gamma F(t) = t^{1-\gamma} F'(t) \tag{21}$$

***Theorem 1.5*** [5, 14] If $F : (0,a) \to \mathbb{R}_\mathcal{F}$ is $\gamma$-differentiable then it is continuous.

Remark 2 If $F : (0,a) \to \mathbb{R}_\mathcal{F}$ is $\gamma$-differentiable and $F^{(\gamma)}$ for all $\gamma \in (0,1]$ is continuous, then we denote $F \in C^1((0,a), \mathbb{R}_\mathcal{F})$.

***Theorem 1.6*** [5, 14] Let $\gamma \in (0,1]$ and if $F, G : (0,a) \to \mathbb{R}_\mathcal{F}$ are $\gamma$-differentiable and $\lambda \in \mathbb{R}$ then

$$T_\gamma(F+G)(t) = T_\gamma(F)(t) + T_\gamma(G)(t) \quad and \quad T_\gamma(\lambda F)(t) = \lambda T_\gamma(F)(t). \tag{22}$$

***Definition 4*** [5] *Let $F \in C((0,\ a),\ \mathbb{R}_\mathcal{F}) \cap L^1((0,\ a),\ \mathbb{R}_\mathcal{F})$, Define the fuzzy fractional.*

*integral for $a \geq 0$ and $\gamma \in (0,1]$.*

$$I_\gamma^a(F)(t) = I_1^a\left(t^{\gamma-1}F\right)(t) = \int_a^t \frac{F}{s^{1-\gamma}}(s)ds, \tag{23}$$

*where the integral is the usual Riemann improper integral.*

***Theorem 1.7*** [5] $T_\gamma I_\gamma^a(F)(t)$, for $t \geq a$, where $F$ is any continuous function in the domain of $I_\gamma^a$.

***Theorem 1.8*** [5] Let $\gamma \in (0,1]$ and $F$ be $\gamma$-differentiable in $(0,a)$ and assume that the conformable derivative $F^{(\gamma)}$ is integrable over $(0,a)$. Then for each $s \in (0,a)$ we have

$$F(s) = F(a) + I_\gamma^a F^{(\gamma)}(t) \tag{24}$$

## 4. Existence and uniqueness solution to fuzzy fractional differential equations

In this section we state the main results of the paper, i.e. we will concern ourselves with the question of the existence theorem of approximate solutions by using the embedding results on fuzzy number space $(\mathbb{R}_{\mathcal{F}}, d)$ and we prove the uniqueness theorem of solution for the Cauchy problem of fuzzy fractional differential equations of order $\gamma \in (0, 1]$.

### 4.1 Solution and its approximate solutions

Assume that $F : (0, a) \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is continuous $C(((0, a) \times \mathbb{R}_{\mathcal{F}}), \mathbb{R}_{\mathcal{F}})$. Consider the fractional initial value problem

$$T_{\gamma}(u)(t) = F(t, u(t)), \quad u(t_0) = u_0, \tag{25}$$

where $u_0 \in \mathbb{R}_{\mathcal{F}}$ and $\gamma \in (0, 1]$.

From Theorems (1.5), (1.7) and (1.8), it immediately follows:

***Theorem 1.9*** A mapping $u : (0, a) \to \mathbb{R}_{\mathcal{F}}$ is a solution to the problem (25) if and only if it is continuous and satisfies the integral equation

$$u(t) = u_0 + \int_{t_0}^{t} s^{\gamma-1} F(s, u(s)) ds \tag{26}$$

for all $t \in (0, a)$ and $\gamma \in (0, 1]$.

In the following we give the relation between a solution and its approximate solutions.

We denote $\Delta_0 = [t_0, t_0 + \theta] \times B(u_0, \mu)$ where $\theta, \mu$ be two positive real numbers $u_0 \in \mathbb{R}_{\mathcal{F}},\ B(u_0, \mu) = \{x \in \mathbb{R}_{\mathcal{F}} | d(u, u_0) \leq \mu\}$.

***Theorem 1.10*** Let $\gamma \in (0, 1]$ and $F \in C(\Delta_0, \mathbb{R}_{\mathcal{F}}),\ \ \eta \in (0, \theta)$, $u_n \in C^1([t_0, t_0 + \eta],\ \ B(u_0, \mu))$ such that

$$ju_n^{(\gamma)}(t) = jF(t, u_n(t)) + B_n(t), \quad u_n(t_0) = u_0, \ \ \|B_n(t)\| \leq \varepsilon_n \tag{27}$$
$$\forall t \in [t_0, t_0 + \eta],\ \ n = 1, 2, \ldots.$$

where $\varepsilon_n > 0,\ \ \varepsilon_n \to 0$, $B_n(t) \in C([t_0, t_0 + \eta], X)$, and $j$ s the isometric embedding from $(\mathbb{R}_{\mathcal{F}}, d)$ onto its range in the Banach space $X$. For each $t \in [t_0, t_0 + \eta]$ there exists an $\beta > 0$ such that the H-differences $u_n(t + \varepsilon t^{1-\gamma}) \ominus u_n(t)$ and $u_n(t) \ominus u_n(t - \varepsilon t^{1-\gamma})$ exist for all $0 \leq \varepsilon < \beta$ and $n = 1, 2, \ldots$. If we have

$$d(u_n(t), u(t)) \to 0 \tag{28}$$

uniform convergence (u.c) for all $t \in [t_0, t_0 + \eta],\ \ n \to \infty$, then $u \in C^1([t_0, t_0 + \eta], B(u_0, \mu))$ and

$$T_{\gamma}(u(t)) = F(t, u(t)), \quad u(t_0) = u_0,\ \ t \in [t_0, t_0 + \eta]. \tag{29}$$

***Proof:*** By (28) we know that $u(t) \in C([t_0, t_0 + \eta], B(u_0, \mu))$. For fixed $t_1 \in [t_0, t_0 + \eta]$ and any $t \in [t_0, t_0 + \eta],\ \ t > t_1,$ denote $\varepsilon = ht_1^{\gamma-1}$ and $\forall\ \ \gamma \in (0, 1]$

$$G(t, n) = \frac{ju_n\left(t_1 + \varepsilon t_1^{1-\gamma}\right) - ju_n(t_1)}{\varepsilon} - jF(t_1, u_n(t_1)) - B_n(t_1). \tag{30}$$

$$= \frac{ju_n(t_1 + h) - ju_n(t_1)}{ht_1^{\gamma-1}} - jF(t_1, u_n(t_1)) - B_n(t_1). \tag{31}$$

$$= t_1^{1-\gamma} \frac{ju_n(t) - ju_n(t_1)}{t - t_1} - jF(t_1, u_n(t_1)) - B_n(t_1). \tag{32}$$

It is well know that

$$\lim_{t\to t_1} G(t,n) = j(u_n)^{(\gamma)}(t_1) - jF(t_1, u_n(t_1)) - B_n(t_1) \tag{33}$$

$$= j(u_n)^{(\gamma)}(t_1) - jF(t_1, u_n(t_1)) - B_n(t_1) = \Theta \in X \tag{34}$$

$$\lim_{n\to\infty} G(t,n) = t_1^{1-\gamma}\frac{ju(t) - ju(t_1)}{t - t_1} - jF(t_1, u(t_1)) \tag{35}$$

From $F \in C^1(\Delta_0, \mathbb{R}_{\mathcal{F}})$, is know that for any $\varepsilon > 0$, there exists $\beta_1 > 0$ such that

$$d(F(t,v), F(t_1, u(t_1))) < \frac{\varepsilon}{4} \tag{36}$$

whenever $t_1 < t < t_1 + \beta_1$ and $d(v, u(t_1)) < \beta_1$ with $v \in B(u_0, \mu)$ Take natural number $N > 0$ such hat

$$\varepsilon_n < \frac{\varepsilon}{4}, d(u_n(t), u(t)) < \frac{\beta_1}{2} \quad \textit{for any} \;\; n > N, \;\; t \in [t_0, t_0 + \eta] \tag{37}$$

Take $\beta > 0$ such that $\beta < \beta_1$ and

$$d(u(t), u(t_1)) < \frac{\beta_1}{2} \tag{38}$$

whenever $t_1 < t < t_1 + \beta$.

By the definition of $G(t,n)$ and (27), we have $\forall\gamma \in (0,1]$

$$ju_n\left(t_1 + \varepsilon t_1^{1-\gamma}\right) - ju_n(t_1) - (\varepsilon)j(u_n)^{(\gamma)}(t_1) = (\varepsilon)jF(t_1, u_n(t_1)) \tag{39}$$

$$t_1^{1-\gamma}\left(ju_n(t) - ju_n(t_1)\right) - (t - t_1)t_1^{1-\gamma}j(u_n)'(t_1) = (t - t_1)jF(t_1, u_n(t_1)) \tag{40}$$

We choose $\psi \in X^*$ such that $\|\psi\| = 1$ and for all $\gamma \in (0,1]$

$$\psi\left(t_1^{1-\gamma}\left(ju_n(t) - ju_n(t_1)\right) - (t - t_1)t_1^{1-\gamma}j(u_n)'(t_1)\right) \tag{41}$$

$$= \|t_1^{1-\gamma}\left(ju_n(t) - ju_n(t_1)\right) - (t - t_1)t_1^{1-\gamma}j(u_n)'(t_1)\| \tag{42}$$

Let $t_1^{1-\gamma}\varphi(t) = t_1^{1-\gamma}\psi\left(ju_n(t)\right) - (t - t_1)t_1^{1-\gamma}j(u_n)'(t_1)$, consequently

$$t_1^{1-\gamma}\varphi'(t) = t_1^{1-\gamma}\psi\left( ju_n'(t)\right) - t_1^{1-\gamma}j(u_n)'(t_1) \tag{43}$$

hence

$$\|t_1^{1-\gamma}\left(ju_n(t) - ju_n(t_1)\right) - (t - t_1)t_1^{1-\gamma}j(u_n)'(t_1)\| \tag{44}$$

$$= t_1^{1-\gamma}(\varphi(t) - \varphi(t_1)) = t_1^{1-\gamma}\varphi'(\hat{t})(t - t_1) \tag{45}$$

$$= \psi\left(t_1^{1-\gamma}\left( ju_n'(\hat{t}) - ju_n'(t_1)\right)\right)(t - t_1) \tag{46}$$

$$\leq \|\psi\|\|t_1^{1-\gamma}\left(ju_n'(\hat{t}) - ju_n'(t_1)\right)\|(t - t_1) \tag{47}$$

$$= \|t_1^{1-\gamma}\left( ju_n'(\hat{t}) - ju_n'(t_1)\right)\|(t - t_1), \tag{48}$$

where $t_1 \leq \hat{t} \leq t$. In view of (40), we have

$$\|G(t,n)\| \leq \|t_1^{1-\gamma}\left( ju_n'(\hat{t}) - ju_n'(t_1)\right)\|, \quad t_1 \leq \hat{t} \leq t. \tag{49}$$

From (37) and (38) we know that

$$d(u(\hat{t}), u(t_1)) < \frac{\beta_1}{2} \tag{50}$$

and

$$d(u_n(\hat{t}), u(t_1)) \leq d(u_n(\hat{t}), u(\hat{t})) + d(u(\hat{t}), u(t_1)) \tag{51}$$

$$< \frac{\beta_1}{2} + \frac{\beta_1}{2} = \beta_1 \tag{52}$$

Hence by (36) and (49) we have for all $\gamma \in (0,1]$.

$$\|G(t,n)\| \leq \|t_1^{1-\gamma}\left(ju_n'(\hat{t}) - ju_n'(t_1)\right)\| \tag{53}$$

$$= \|jF(\hat{t}, u_n(\hat{t})) + B_n(\hat{t}) - jF(t_1, u_n(t_1)) - B_n(t_1)\| \tag{54}$$

$$\leq \|jF(\hat{t}, u_n(\hat{t})) - jF(t_1, u(t_1))\| \tag{55}$$

$$+\|jF(t_1, u(t_1)) - jF(t_1, u_n(t_1))\| + 2\varepsilon_n \tag{56}$$

$$\leq d(jF(\hat{t}, u_n(\hat{t})) - jF(t_1, u(t_1))) \tag{57}$$

$$+d(jF(t_1, u(t_1)) - jF(t_1, u_n(t_1))) + 2\varepsilon_n \tag{58}$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + 2\varepsilon_n < \varepsilon \tag{59}$$

whenever $n > N$ and $t_1 < t < t_1 + \beta$.
Let $n \to \infty$, and applying (35), we have

$$\|t_1^{1-\gamma}\frac{ju(t) - ju(t_1)}{t - t_1} - jF(t_1, u(t_1))\| \leq \varepsilon, \quad t_1 < t < t_1 + \beta. \tag{60}$$

On the other hand, from the assumption of Theorem (1.9), there exists an $\beta(t_1) \in (0, \beta)$ such that the H-differences $u_n(t) \ominus u_n(t_1)$ exist for all $t \in [t_1, t_1 + \beta(t_1)]$ and $n = 1, 2, \ldots$.

Now let $v_n(t) = u_n(t) \ominus u_n(t_1)$ we verify that the fuzzy number-valued sequence $\{v_n(t)\}$ uniformly converges on $[t_1, t_1 + \beta(t_1)]$. In fact, from the assumption $d(u_n(t), u(t)) \to 0$ u.c. for all $t \in [t_0, t_0 + \eta]$, we know

$$d(v_n(t), v_m(t)) = d(v_n(t) + u_n(t_1), v_m(t) + u_n(t_1)) \tag{61}$$

$$\leq d(u_n(t), u_m(t)) + d(u_m(t), v_m(t) + u_n(t_1)) \tag{62}$$

$$= d(u_n(t), u_m(t)) + d(v_m(t) + u_m(t_1), v_m(t) + u_n(t_1)) \tag{63}$$

$$= d(u_n(t), u_m(t)) + d(u_m(t_1), u_n(t_1)) \tag{64}$$

$$\to \quad u.c \ \ \forall t \in [t_1, t_1 + \beta(t_1)] \ \ n, m \to \infty. \tag{65}$$

Since $(\mathbb{R}_{\mathcal{F}}, d)$ is complete, there exists a fuzzy number-valued mapping $v(t)$ such that $\{v_n(t)\}$ u.c to $v(t)$ on $[t_1, t_1 + \beta(t_1)]$ as $n \to \infty$.

In addition, we have

$$d(u(t_1) + v(t), u(t)) \leq d(u(t_1) + v(t), u_n(t_1 + v_n(t_1)) + d(u_n(t_1 + v_n(t), u(t)) \tag{66}$$

$$\leq d(u(t_1) + v(t), u(t_1) + v_n(t)) \tag{67}$$

$$+ d(u(t_1) + v_n(t), u_n(t_1) + v_n(t)) + d(u_n(t), u(t)) \tag{68}$$

$$= d(v_n(t), u(t)) + d(u_n(t_1), u(t_1)) + d(u_n(t), u(t)) \tag{69}$$

$$\forall t \in [t_1, t_1 + \beta(t_1)].$$

Let $n \to \infty$. It follows that

$$u(t_1) + v(t) \equiv u(t) \quad \textit{for all} \;\; t \in [t_1, t_1 + \beta(t_1)]. \tag{70}$$

Hence the H-difference $u(t) \ominus u(t_1)$ exist for all $t \in [t_1, t_1 + \beta(t_1)]$.
Thus from (60) we have for all $\gamma \in (0, 1]$.

$$d\left(\frac{u\left(t_1 + t_1^{1-\gamma}\varepsilon\right) \ominus u(t_1)}{\varepsilon}, F(t_1, u(t_1))\right) \le \varepsilon, \quad t \in [t_1, t_1 + \beta(t_1)]. \tag{71}$$

So, $\lim_{\varepsilon \to 0^+} u\left(t_1 + t_1^{1-\gamma}\varepsilon\right) \ominus u(t_1)/\varepsilon = F(t_1, u(t_1))$. Similarty, we have

$$\lim_{\varepsilon \to 0^-} \frac{u\left(t_1 + t_1^{1-\gamma}\varepsilon\right) \ominus u(t_1)}{\varepsilon} = F(t_1, u(t_1)).$$

Hence $u^{(\gamma)}(t_1)$ exists and

$$u^{(\gamma)}(t_1) = F(t_1, u(t_1)). \tag{72}$$

from $t_1 \in [t_0, t_0 + \eta]$ is arbitrary, we know that Eq. (29) holds true and $u \in C^1([t_0, t_0 + \eta], B(u_0, \mu))$. The proof is concluded.

***Lemma 2*** For all $t \in [t_0, t_0 + \eta]$, $n = 1, 2, \ldots$ and $\gamma \in (0, 1]$.

If we replace Eq. (27) by

$$ju_{n+1}(t) = jF(t, u_n(t)) + B_n(t), \quad u_n(t_0) = u_0, \;\; \|B_n(t)\| \le \varepsilon_n, \tag{73}$$

retain other assumptions, then the conclusions also hold true.

***Proof:*** This is completely similar to the proof of Theorem (1.10), hence itis omitted here.

### 4.2 Uniqueness solution

In this section, by using existence theorom of approximate solutions, and the embedding results on fuzzy number space $(\mathbb{R}_{\mathcal{F}}, d)$, we give the existence and uniqueness theorem for the Cauchy problem of the fuzzy fractional differential equations of order $\gamma$.

***Theorem 1.11***

i. Let $F \in C(\Delta_0, \mathbb{R}_{\mathcal{F}})$ and $d\left(F(t, u), \hat{0}\right) \le \sigma$ for all $(t, u) \in \Delta_0$.

ii. $G \in C([t_0, t_0 + \theta] \times [0, \mu], \mathbb{R})$, $G(t, 0) \equiv 0$, and $0 \le G(t, y) \le \sigma_1$, for all $t \in [t_0, t_0 + \theta], 0 \le y \le \mu$ such that $G(t, y)$ is noncreasing on $y$ the fractional initial value problem

$$T_\gamma y(t) = G(t, y(t)), \quad y(t_0) = 0 \tag{74}$$

has only the solution $y(t) \equiv 0$ on $[t_0, t_0 + \theta]$.

iii. $d(F(t,u), F(t,v)) \leq G(t, d(u,v))$ for all $(t,u), (t,v) \in \Delta_0$, and $d(u,v) \leq \mu$.

Then the Cauchy problem (29) has unique solution $u \in C^1([t_0, t_0 + \eta], B(u_0, \mu))$ on $[t_0, t_0 + \eta]$, where $\eta = \min\{\theta, \mu/\sigma, \mu/\sigma_1\}$, and the successive iterations

$$u_{n+1}(t) = u_0 + \int_{t_0}^{t} s^{\gamma-1} F(s, u_n(s)) ds \tag{75}$$

uniformly converge to $u(t)$ on $[t_0, t_0 + \eta]$.

***Proof:*** In the proof of Theorem 4.1 in [15], taking the conformable derivative $u^{(\gamma)}$ for all $\gamma \in (0,1]$, using theorem (1.4) and properties (10), then we obtain the proof of Theorem (1.11).

***Example 1*** Let $L > 0$ is a constant, taking $G(t,y) = Ly$ in the proof of Theorem (4.2), then obtain the proof of Corollary 4.1 in [15] where $\sigma_1 = L\mu$, hence $\eta = \min\{\theta, \mu/\sigma, 1/L\}$. Then the Cauchy problem (29) has unique solution $u \in C^1([t_0, t_0 + \eta], B(\Delta_0, \mu))$, and the successive iterations (75) uniformly converge to $u(t)$ on $[t_0, t_0 + \eta]$.

## 5. Conclusion

In this work, we introduce the concept of conformable differentiability for fuzzy mappings, enlarging the class of $\gamma$-differentiable fuzzy mappings where $\gamma \in (0,1]$. Subsequently, by using the $\gamma$-differentiable and embedding theorem, we study the Cauchy problem of fuzzy fractional differential equations for the fuzzy valued mappings of a real variable. The advantage of the $\gamma$-differentiability being also practically applicable, and we can calculate by this derivative the product of two functions because all fractional derivatives do not satisfy see [4].

On the other hand, we show and prove the relation between a solution and its approximate solutions to the Cauchy problem of the fuzzy fractional differential equation, and the existence and uniqueness theorem for a solution to the problem (2) are proved.

For further research, we propose to extend the results of the present paper and to combine them the results in citeref for fuzzy conformable fractional differential equations.

## Conflict of interest

The authors declare no conflict of interest.

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